JUMP PROCESSES AND THEIR MARTINGALES

by

Mark H.A. Davis

1. Introduction.

A jump process, as defined here, is a right-continuous piecewise-constant stochastic process (x_t) taking values in a Polish space X. We assume that the process has discontinuities at an increasing sequence of isolated times (T_k) and is killed at time $T_\infty := lim_k T_k$. Thus a sample function of the process is specified by giving a random varable Z_0 and a sequence (S_k, Z_k) for $k = 1, 2, ..$ of random variables with $S_k \in \mathbf{R}_+$ and $Z_k \in E$ and defining $T_0 = 0, T_k = T_{k-1} + S_k$ and $X_t = Z_k$ for $t \in [T_k, T_{k+1}[$, $X_t = \Delta_\infty$ for $t \geq T_\infty$, where Δ_∞ is an isolated "cemetary state".

Jump processes have the "martingale representation property": all local martingales with respect to the natural filtration of (X_t) can be expressed as "stochastic integrals" with respect to a certain family of martingale measures associated with the process. In this paper we give a streamlined proof of this result. The result is analogous to Ito's famous theorem on the representation of Brownian local martingales, but with two differences: it is *simpler* in that no special definition of the stochastic integral is required (all integrals are Stieltjes integrals evaluated separately for each sample path of the process), but *more complicated*, in that a whole family of "elementary martingales" is required, not just a single one as in the Brownian case (i.e. the Brownian motion itself).

Jump process Martingales were studied in a series of papers in the 1970s: Boel, Varaiya and Wong (1975), Jacod (1975), Chou and Meyer (1975), Davis (1976), Elliott (1976). Some of this material, but not all of it, appears in the textbooks Brémaud (1980), Elliott (1982). Here we follow the argument of Davis (1976) closely, but a number of technical improvements make the presentation more self-contained. Specifically, some of the arguments given by Brémaud (1980) for "right-constant" processes enable us to show rather directly in §3 that the stopped σ-field $\mathcal{F}_T$ for a stopping time T is essentially the σ-field generated by the process up to the stopping time. Also, we use systematically a constructive definition of "predictability" (given in §4), avoiding the need to introduce "predictable processes" as defined in the "théorie générale des processes" This is possible bacause our filtration is generated in a very particular way.

The jump process is formally defined in §2 below. §§3,4 concern the structure of stopping times and "predictable processes" (our definition). The martingale repesentation results are stated and proved in §§5, 6. As in Chou and Meyer (1975), Davis (1976), the approach is to study the elementary "single jump" process first and to use this as a building block for the general case.

2. DEFINITION OF THE JUMP PROCESS

As discussed above, the jump process (x_t) will take values in a Polish (complete separable metric) space X together with an additional isolated point Δ_∞. The sample path takes the form

$$x_t = z_o I_{t<T_1} + \sum_{i=1}^{\infty} Z_i I_{T_i \le t < T_{i+1}} + \Delta_\infty I_{t \ge T_\infty}$$

where z_0 is a non-random point in x, $Z_1, Z_2, ..$ are x-valued random variables and $0 < T_1 < T_2 ..$ are random times with $T_\infty := lim_k T_k$. It is possible that $T_k = \infty$ for some k. We will assume that $P[Z_k = Z_{k-1}] = 0$ for all k, so that the process really does "jump" at the "jump times" T_k. We can define the jump process on a canonical pace, as follows. Let

$$Y = (\mathbf{R}_+ \times X) \cup \{\Delta\}$$

where Δ is an isolated point and let $\mathcal{Y}$ denote the Borel sets of Y. Define $\Omega_i = \Pi_{k=1}^{i} \Omega$, $\Omega = \Pi_{k=1}^{\infty} Y_k$, $\mathcal{F}^{i,0} = \sigma\{\Pi_{k=1}^{i} \mathcal{Y}_k\}$ and $\mathcal{F}^\circ = \sigma\{\Pi_{k=1}^{\infty} \mathcal{Y}_k\}$. Let $\xi_k : \Omega \to Y_k$ denote the coordinate mapping and let $\xi_k(\omega) = (S_k(\omega), Z_k(\omega))$ when $\xi_k(\omega) \in \mathbf{R}_+ \times x$ (otherwise, $\xi_i(\omega) = \Delta$). Let $\omega_k(\omega) = (\xi_1(\omega), .., \xi_k(\omega))$. Now let

$$T_k(\omega) := \begin{cases} \sum_{i=1}^{k} S_i(\omega) & \text{if} \quad \xi_i(\omega) \neq \Delta, i = 1, .., k \\ \infty & \text{if} \quad \xi_i(\omega) = \Delta \text{ for some } i = 1, .., k \end{cases}$$

$$T_\infty(\omega) := \lim_k T_k(\omega)$$

and define the sample path $x_t(\omega)$ for $t \in \mathbf{R}_+$ by

$$x_t(\omega) = \begin{cases} z_o & t < T_1(\omega) \\ Z_k & T_k(\omega) \le t < T_{k+1}(\omega) \\ \Delta_\infty & t \ge T_\infty(\omega). \end{cases}$$

Here $z_0 \in Z$ is fixed and Δ_∞ is a point isolated from X. The natural filtration of the process (x_t) in Ω is

$$\mathcal{F}_t^\circ := \sigma\{x_s(\cdot), s \le t\}.$$

A probability measure on Ω is defined by giving the following family of conditional distribution functions: μ^1 is a probability measure on Y such that

(1) $\qquad \mu^1((\{0\} \times X) \cup (\mathbf{R}_+ \times \{z_0\})) = 0,$

and for $k = 2, 3, ..$ $\qquad \mu^k : \Omega_{k-1} \times \mathcal{Y} \to [0,1]$ is a transition measure satisfying

(i) $\mu^k(\cdot; \Gamma)$ is measurable for each $\Gamma \in \mathcal{Y}$

(ii) $\mu^k(\omega_{k-1}(\omega); \cdot)$ is a probability measure for each $\omega \in \Omega$

(iii) $\mu^k(\omega_{k-1}(\omega); (\{0\} \times X) \cup \{\mathbf{R}_+ \times Z_{k-1}(\omega)\})) = 0$ for each $\omega \in \Omega$

(iv) $\mu^k(\omega_{k-1}(\omega); \{\Delta\}) = 1$ if $\xi_i(\omega) = \Delta$ for some $i \leq k-1$.

Then P is the unique probability measure on $(\Omega, \mathcal{F}^\circ)$ such that for each k and bounded measurable function f on Ω_k

$$\int_\Omega f(\xi_1(\omega), .., \xi_k(\omega)) P(d\omega)$$
$$= \int_{Y_1} \cdot\cdot \int_{Y_k} f(\xi_1, .., \xi_k) \mu^k(\xi_1, .., \xi_{k-1}; d\xi_k) \mu^{k-1}(\xi_1, .., \xi_{k-1}; d\xi_{k-1}) ... \mu^1(d\xi_1)$$

Note from (iii) that, with probability one, $T_1 > 0, T_k > T_{k-1}$ and $Z_k \neq Z_{k-1}$. Also, (iv) implies that $\xi_i = \Delta$ for all $i \geq k := min\{j : \xi_j(\omega) = \Delta\}$ and we interpret this as saying that $T_k(\omega) = \infty$. We now define $\mathcal{F}_t[\mathcal{F}^k, \mathcal{F}]$ to be the σ-field $\mathcal{F}_t^\circ[\mathcal{F}^{k,0}, \mathcal{F}^\circ]$ completed with all P-null sets of $\mathcal{F}^\circ$. Let $\mathcal{F}$ denote the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbf{R}_+}$

(2) Lemma

a) $T_k, k = 1, 2, ..$ and T_∞ are $\mathcal{F}$ – stopping times

b) $\mathcal{F} = \mathcal{F}_\infty := \vee_{t \in \mathbf{R}_+} \mathcal{F}_t$.

Proof:

a) Let $N_t := \sum_i I_{t \geq T_i}$. In view of $(1)(iii), (_t)$ is clearly an $\mathcal{F}_t$-adapted process, and $(T_i \leq t) = (N_t \geq i)$. Also $(T_\infty \leq t) = \cap_i (T_i \leq t)$.

b) By definition $\mathcal{F}^\circ = \sigma\{\xi_i, i = 1, 2, ..\}$ so that $\mathcal{F}_\infty \subset \mathcal{F}$. For the converse it suffices to show that ξ_i is $\mathcal{F}_\infty$-measurable for each i. Now $(\xi_i = \Delta) = \cap_n (T_i > n)$ while $(S_i \leq t, Z_i \in A) = (T_i \leq T_{i-1} + t) \cap (T_{i-1} < \infty) \cap (Z_i \in A) \in \mathcal{F}_\infty$. This completes the proof. $\square$

A.3. Structure of stopping times and stopped σ-fields

Recall that for any $\mathcal{F}_t$-stopping time T, the *stopped σ-field* $\mathcal{F}_T$ is defined as

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap (T \leq t) \in \mathcal{F}_t \quad \text{for all} \quad t \in \mathbf{R}_+\}$$

We need a more explicit characterization of $\mathcal{F}_T$, and this is given by the following theorem, which also shows that the filtration $\mathcal{F}$ is right-continuous.

(1) Theorem

(a) For any $t \in \mathbf{R}_+$ we have $\mathcal{F}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$.
(b) For any stopping time T we have

$$\mathcal{F}_t = \sigma\{x_{s\wedge T}, s \in \mathbf{R}_+\}.$$

(c) For each $k = 1, 2, ..$

$$\mathcal{F}_{T_k} = \sigma\{\xi_1, .., \xi_k\} = \mathcal{F}^k \times \Pi_{i=k+1}^{\infty} Y_i,$$

i.e. $A \in \mathcal{F}_{T_k}$ if and only if $A = A' \times \Pi_{i=k+1}^{\infty} Y_i$ for some $A' \in \mathcal{F}^k$

Parts (a) and (b) of the theorem are true for any right-constant process, and it is expeditious to prove them in this generality. Thus let $(Y_t)_{t\in\mathbf{R}_+}$ be an X-valued stochastic process with right-continous sample paths defined on some probability space $(B, \mathcal{A}, m)$ and let $(\mathcal{Y}_t)$ be the natural filtration of (Y_t) completed as usual with all m-rull sets of $\mathcal{A}$. (Y_t) is ***right-constant*** if for each $(t, \beta) \in R_+ \times B$ there exists $\epsilon(t, \beta) > 0$ such that $Y_{t+\delta}(\beta) = Y_t(\beta)$ for $\delta \in [0, \epsilon(r, \beta)]$

(2) Theorem [Brémaud (1981), Appendix A2]

Suppose (Y_t) is a right-constant process as described above. Then

(a) $\mathcal{Y}_t = \mathcal{Y}_{t+} := \bigcap_{\epsilon>0} \mathcal{Y}_{t+\epsilon}$ for each $t \in \mathbf{R}_+$.

(b) $\mathcal{Y}_S = \sigma\{Y_{s\wedge S}, s \in \mathbf{R}_+\}$ for each $\mathcal{Y}_t$-stopping time S.

Proof

(a) It suffices to show that if $A \in \mathcal{A}$ is a set which is in $\mathcal{Y}_{t+2^{-k}}$ for all k, then A is in $\mathcal{Y}_t$. In view ofthe fact that Y_t has right-continous sample paths and the supposition that $A \in \mathcal{Y}_{t+2^{-k}}$ we can write the indicator function of A in the form

$$I_A = \Phi_k(Y_s, s \in Q \cap [0, t + 2^{-k}])$$

where Q denotes the set of rational numbers. Now define

$$\eta_k := \Phi_n'(Y_s, s \in Q \cap [0, t]) := \Phi_n(Y_s', s \in Q \cap [0, t + 2^{-k}])$$

where

$$Y'_s = \begin{cases} Y_s & s \in [0, t[\\ Y_t & s \in [t, t + 2^{-k}[\end{cases}$$

and

$$B_k = \{\beta \in B : Y_{t+s}(\beta) = Y_t(\beta) \quad \delta \leq [0, 2^{-k}]\}.$$

Then evidently η_k is $\mathcal{Y}_t$-measurable, $B_k \uparrow B$ and

$$I_{A \cap B_k} = \eta_k I_{B_k}.$$

It follows that $I_A = lim\ inf \eta_k$, and hence that $A \in \mathcal{Y}_t$.

(b) Let S be a $\mathcal{Y}_t$ stopping time and

$$\mathcal{G} = \sigma\{Y_{s \wedge S}, s \geq 0\}.$$

If $A := (Y_{s \wedge S} \in G)$ for some $G \in \mathcal{B}(X)$ then clearly $A \cap (S \leq t) \in \mathcal{Y}_t$. Hence $\mathcal{G} \subset \mathcal{Y}_S$. For the converse, suppose first that S takes values $0 \leq a_1 < a_2 .. \leq \infty$. Then any $A \in \mathcal{Y}_S$ can be written $A = U_i A_i$ where $A_i = A \cap (S = a_i) = \mathcal{Y}_{a_i}$. As above we can write

$$I_{A_i} = \Phi_i(Y_t, t \in Q \cap [0, a_i]),$$

and since $S = a_o$ on A_i this is the same as

$$I_{A_i} = \Phi_i(Y_{t \cap S}, t \in Q \cap [0, a_i]).$$

Thus $A_i \in \mathcal{G}$ and hence $A \in \mathcal{G}$.

For the general case, let

$$S_k = \sum_{i=1}^{\infty} \frac{i}{2^k} I_{((i-1)2^{-k} \leq S < i2^{-k})} + \infty I_{(S=\infty)}$$

then S_k is countably-valued, and $S_k \downarrow S$, so that $\mathcal{Y}_S \subset \mathcal{Y}_{S_k}$. From the above $\mathcal{Y}_{S_k} = \sigma\{Y_{s \wedge S_k}, s \geq 0\}$. Let B_k be defined as above but with S replacing t. We then have

$$\mathcal{Y}_{S_k} \cap B_k = \sigma\{Y_{s \wedge S}, s \geq 0\} \cap B_k$$

Where, for a σ-field $\mathcal{H}$, $\mathcal{H} \cap B_k = \{H \cap B_k : H \in \mathcal{H}\}$. Thus if $A \in \mathcal{Y}_S \subset \mathcal{Y}_{S_{k+n}}$ then there exists $G_{k+n} \in \mathcal{G}$ such that $A \cap B_{k+n} = G_{k+n} \cap B_{k+n}$. Intersecting each side with B_k gives $A \cap B_k = G_{k+n} \cap B_k$ and it follows that $A \cap B_k = G \cap B_k$ where $G := \liminf G_n \in \mathcal{G}$. Since $B_k \uparrow B$ this shows that $A = G \in \mathcal{G}$ and hence that $\mathcal{Y}_S \subset \mathcal{G}$. □

Proof of Theorem (1):
Parts (a) and (b) are special cases of Theorem (2). For part (c) denote

$$\mathcal{H} = \mathcal{F}^k \times \Pi_{i=k+1}^{\infty} Y_i = \sigma\{\xi_i, \xi_2, .., \xi_k\}.$$

From part (b)

$$\mathcal{F}_{T_k} = \sigma\{x_{t\wedge T_k}, t \geq 0\} =: \mathcal{G}$$

But there is a 1-1 correspondence between $(\xi_1, .., \xi_k)$ and $(x_{t\wedge T_k}, t \geq 0)$, and hence $\mathcal{H} = \mathcal{G}$. $\square$

We shall also need the following result, giving a very precise description of the class of $\mathcal{F}_t$ stopping times.

(3) Theorem
Let Ω be a stopping time of the jump process natural filtration $\mathcal{F}_t$. Then there exists a constant s_1 and functions $s_k : \Omega_{k-1} \to \mathbf{R}_+$ for $k = 2, 3..$ such that

$$\tau I_{(\tau \leq T_1)} = (s_1 \wedge T_1) I_{(\tau \leq T_1)}$$

and for $k = 2, 3..$

$$\tau I_{(T_{k-1} < \tau \leq T_k)} = ((T_{k-1} + s_k(\xi_{k-1})) \wedge T_k) I_{(T_{k-1} < \tau \leq T_k)}. \tag{4}$$

(5) Remark: An equivalent, and simpler, statement is: there exist $\mathcal{F}_{T_{k-1}}$-measurable random variables τ_k such that $\tau I_{(\tau < T_k)} = \tau_k I_{(\tau < T_k)}$. However, the more explicit form (4) is what we need in applications.

To prove Theorem (3) we first consider a simple situation in which there is only one jump (the 'single -jump' process is analyzed in more detail in §A.4.). Thus, let $(\Xi, \mathcal{S})$ be a measurable space and z_0 be a measurable function mapping Ξ into X (where X is as above). Define $\Omega = (\Xi \times \mathbf{R}_+ \times X) \setminus \{(\xi, t, z) \in \Xi \times \mathbf{R}_+ \times X : z = z_0(\xi)\}$. For $\omega = (\xi, t, z) \in \Omega$, denote $\xi(\omega) = \xi, T(\omega) = t, Z(\omega) = z$. For $t \in \mathbf{R}_+$ define

$$x_t(\omega) = \begin{cases} z_o(\xi), & t < T(\omega) \\ Z(\omega), & t \geq T(\omega). \end{cases}$$

Now let $\mathcal{H}_t$ be the 'natural filtration' in Ω of the 'process' X_t, defined by $\mathcal{H}_0 = \mathcal{S} \times \mathbf{R}_+ \times X, \mathcal{H}_t = \mathcal{H}_0 \vee \sigma\{x_s, s \leq t\}$. Then it is easy to see that

$$\mathcal{H}_t = \Omega \cap (\mathcal{B}(\Xi \times [0, t] \times X) \cup \mathcal{B}(\Xi) \times]t, \infty[\times X). \tag{6}$$

(7) Lemma If τ is an $\mathcal{H}_\tau$-stopping time then

$$\tau \wedge T = t_1(\xi) \wedge T \tag{8}$$

for some measurable function $t_1 : \Xi \to \mathbf{R}_+$.

Proof First, suppose that τ takes one of only a countable number of values $0 \leq a_1 < a_2..$ Let $A_i = \{\omega : \tau(\omega) = a_i\}$. Then $A_i \in \mathcal{H}_{a_i}$, and hence in view of (6),
$A_i = A_i^1 \cup (A_i^2 \times]t, \infty[\times X)$ where $A_i^1 \in \Omega \wedge \mathcal{B}(\Xi \times [0,t] \times X), A_i^2 \in \mathcal{B}(\xi)$. The A_i are disjoint and partition Ω. Now

$$(\tau < T) = \cup_i (\tau < T) \cap A_i = \cup_i (a_i < T) \cap A_i = \cup_i A_i^2 \times]t, \infty[\times X$$

Define

$$h(\xi) = \sum_i a_i I_{A_i^2}(\xi);$$

then $a_i = h(\xi) = \tau(\omega)$ on $A_i \cap (T > \tau)$, so that (8) holds with $t_1(\xi) = h(\xi)$.

For a general stopping time τ, define for $n = 1, 2..$,

$$\tau_n = \sum_{n=1}^{\infty} \frac{k}{2^n} I_{(\frac{k-1}{2^n} < \tau \leq \frac{k}{2^n})}.$$

Then τ_n is a countably-valued stopping time, $\tau_n \geq \tau$ and $\downarrow \tau$ as $n \to \infty$. Let $h_n(\xi)$ be the corresponding sequence of functions as above and define $t_1(\xi) = \liminf_n h_n(\xi)$. Then t_1 is measurable and for $\omega \in (\tau < T)$ there is a number n_0 such that $\tau_n(\omega) < T(\omega)$ for $n > n_0$, so that $\tau_n(\omega) = h_n(\xi)$ for $n > n_0$. Thus $\tau(\omega) = h(\xi)$. □

Proof of Theorem (3). If τ is a stopping time of $\mathcal{F}_t$ then $\tau' := (\tau \wedge T_k - T_{k-1}) \vee 0$ is a stopping time of $\mathcal{H}_t := \mathcal{F}_{(T_{k-1}+t)\wedge T_k}$. Indeed,

$$(\tau' \leq t) = (\tau \leq T_{k-1}) \cup ((\tau > T_{k-1}) \cap (\tau \wedge T_k \leq (t + T_{k-1}) \wedge T_k))$$

and this is an $\mathcal{H}_t$-set since $(\tau \leq T_{k-1}) \in \mathcal{F}_{t_{k-1}} \subset \mathcal{H}_t$. But from Theorem (3.1)(b),(c) we know that $\mathcal{H}_\tau = \mathcal{F}_{t_{k-1}} \vee \sigma\{X_{(s+T_{k-1})\wedge T_k}, s \in [0,t]\}$ and $\mathcal{F}_{T_{k-1}} = \sigma\{\xi_1, .., \xi_{k-1}\}$, so that applying Lemma (7) we conclude that

$$\tau' \wedge (T_k - T_{k-1}) = s_k(\xi_1, .., \xi_{k-1})$$

for some measurable function $s_k : \Omega_{k-1} \to \mathbf{R}_+$. Hence

$$\tau \wedge T_k = \tau \wedge T_{k-1} I_{(\tau \geq T_{k-1})} + (T_{k-1} + \tau') I_{(\tau > T_{k-1})}$$
$$= \tau \wedge T_{k-1} I_{(\tau \leq T_{k-1})} + (T_{k-1} + s_k) I_{\tau > T_{k-1}}.$$

this completes the proof. □

A.4. Predictability

The concept of predictability was introduced (not originally under that name) by P.A. Meyer to obtain uniqueness in the decomposition of a submartingale into the sum of a martingale and an increasing process. Consider for example a Poisson process $N_t = \sum_i I_{t \geq T_i}$ where T_0 and $(T_k - T_{k-1}), k = 1, 2..$ are i.i.d random variables with $P[T_k - T_{k-1} > t] = e^{-\lambda t}$. Then $EN_t = \lambda t$ and it is easy to show that $M_t := N_t - \lambda t$ is a martingale (this is a special case of Proposition (6.1) below.) We call λt the *compensator* of N_t. Since N_t is an increasing process ($N_t \geq N_s$ for $t \leq s$) it is certainly a submartingale. We can therefore decompose it into the sum of a martingale and an increasing process in at least two ways, namely $N_t = M_t + \lambda t$ and $N_t = 0 + N_t$. In order to rule out the second, trivial, decomposition we must place some restriction on the class of increasing processes we are prepared to consider as compensators. In this case the process λt is both continuous and deterministic. But it is easy to construct examples where no continous or deterministic compensator exists (this will be evident in §5 below) and predictability is just the right requirement to secure both existence and uniqueness in a general context. It is, however, widely regarded as a somewhat arcane concept the intuitive significance of which is not easy to grasp (the reader can consult Elliott (1982) §5 for a clear account). Fortunately, we do not need it. The only filtrations considered in this paper are those associated with stochastic jump processes or, equivalently, piecewise-deterministic processes, and for these filtrations a constructive definition is possible which is equivalent to the general definition when the latter is specialized to the jump process case. We will not demonstrate the equivalence here; a proof can be found in Boel, Varaiya and Wong (1975).

Let $\mathcal{F}_t$ be the natural filtration of a jump process (x_t) with jump times $T_1, T_2..$ as defined in §A.2.

(1) Definition A stochastic process $\phi(t, \omega)$ is *predictable* if there exist measurable functions $\phi_1 : \mathbf{R}_+ \to \mathbf{R}$, $\phi_k : \mathbf{R}_+ \times \Omega_{k-1} \to \mathbf{R}$, $k = 2, 3..$ and $\phi_\infty : \mathbf{R}_+ \times \Omega \to \mathbf{R}$ such that

$$\phi(t,\omega) = \phi_1(t)I_{0\le t\le T_1} + \sum_{k=2}^{\infty} \phi_k(t,\omega_{k-1})I_{(T_{k-1}<t\le T_k)} \tag{2}$$
$$+ \phi_\infty(t,\omega)I_{(t\ge T_\infty)}$$

The key point here is that $\phi(t) = \phi_k(t)$ for t up to *and including* T_k.

(3) Definition A stopping time T is *predictable* if the process $I_{(t\ge T)}$ is predictable.

(4) Proposition T_∞ is a predictable stopping time
Proof $I_{t\ge T_\infty}$ has the representation (2) with $\phi_k = 0, k < \infty$, and $\phi_\infty(t,\omega) = 1$.

□

Any deterministic process is predictable so, returning to the Poisson process example, the decomposition $N_t = M_\tau + \lambda t$ does give N_t as the sum of a martingale and a predictable process. However this is not true of the decomposition $N_t = 0 + N_t$, as a consequence of the following theorem, the main result of this section.

(5) Theorem If a process $\phi(t,\omega)$ is $\mathcal{F}_t$-predictable and is a uniformly-integrable martingale with $\phi(0,\omega) = 0$, then $\phi(t,\omega) = 0$ for all t, a.s.

Proof Apply the optional sampling theorem to the stopping times $s\wedge T_1, t\wedge T_1$ with $s \le t$. Then

$$\phi(s\wedge T_1,\omega) = E[\phi(t\wedge T_1,\omega)|\mathcal{F}_{s\wedge T_1}] \quad \text{a.s.},$$

and, because ϕ is predictable, $\phi(t,\omega) = \phi_1(t)$ for some non-random function ϕ_1 on the set $(t \le T_1)$. If F denotes the survivor function of T_1, the above conditional expectation is given by

$$I_{(T_1\le s)}\phi_1(T_1) + I_{(T_1>s)}\left(\frac{F(t)}{F(s)}\phi_1(t) - \frac{1}{F(s)}\int_{]s,t]} \phi_1(u)dF(u)\right).$$

Thus on the set $(T_1 > s)$ we have

$$\phi_1(s) = \frac{F(t)}{F(s)}\phi_1(t) - \frac{1}{F(s)}\int_{]s,t]} \phi_1(u)dF(u). \tag{6}$$

Let $y(t) = \phi_1(t)F(t), dG(t) = dF(t)/F(t)$. Then (6) is equivalent to

$$y(t) - y(s) = \int_{]s,t]} y(u)dG(u), \quad y(0) = 0$$

and $G(u)$ has bounded variation on any interval $[0,t]$ such that $t < c := \inf\{t : F(t) = 0\}$. From Lemma 13.4 of Elliott (1982), the unique (locally bounded) solution to this equation is $y(t) = 0$. Hence $\phi_1(t) = 0$ for $t \in [0,c[$. If $c < \infty$ and $F(c-) = 0$ then can take $\phi_1(t) = 0$ for all $t \geq 0$ since $P[T_1 \geq c] = 0$. If $c < \infty$ and $P(T_1 = c) = F(c-) > 0$, apply the optional sampling theorem to the stopping times $0, T_1$. Since $\phi_1(t) = 0, t < c$ we see that $0 = E[\phi_1(T_1)] = \phi_1(c)F(c-)$ and hence that $\phi_1(c) = 0$. We can now apply the same argument inductively on stochastic intervals $]T_{k-1}, T_k]$ to show that $\phi_k = 0, k = 2, 3..$

□

(7) Corollary The Poisson process N_t is not predictable, for any $\lambda > 0$.

Proof: We know that $M_t := N_t - \lambda t$ is a martingale, and since N_t is increasing it is clear that $M_{t\wedge n}$ is a uniiformly integrable martingale for any $n > 0$. If N_t is predictable then $M_{t\wedge n}$ is a predictable martingale, and hence equal to 0 by theorem (5). This is a contradiction unless $\lambda = 0$.

□

The same reasoning shows that λt is the *unique* predictable compensator of N_t, since if ϕ_τ were another then $M'_t = N_\tau - \phi_t$ would be a martingale and $M_t - M'_t = \phi_\tau - \lambda t$ a predictable martingale; hence $\phi_\tau = \lambda t$. Predictable stopping times were defined at (3) above. The following more explicit characterization is easily obtained from the definition of predictability, and complements the description of an *arbitrary* stopping time given by Theorem (3.3)

(8) Proposition Let T be a predictable $\mathcal{F}_t$-stopping time. Then there exist a constant $s_1, \mathcal{F}_{T_{k-1}}$-measurable random variables s_k for $k = 2, 3..$ and an $\mathcal{F}$-measurable random variable s_∞, all taking values in $[0, \infty]$ such that, with $T_0 = 0, T = T_{p-1} + s_p$ where $p = \inf\{k : T_{k-1} + s_k \leq T_k\}$ or $T = T_\infty + s_\infty$ if the set $\{..\}$ is empty.

(9) Example $T := T_7 + 1$ is a predictable time, with $s_k = \infty, k \leq 7, s_8 = 1, s_k = (1 - (T_{k-1} - T_7)) \vee 0, 9 \leq k \leq \infty$.

A.5 The single jump process

To analyse jump process martingales, we begin by studying in detail the "single-jump" case; the original process can then be treated by decomposing it into a sum of single-jump processes starting at the successive jump times of the original process.

Formally, the single jump process is the special case of the jump process definition in §A.2

in which $p^2(\omega_1(\omega); \{\Delta\}) = 1$, but it is more convenient to define it on its own canonical space $(Y, \mathcal{Y})$ equipped with a probablility measure for staisfying (3.1); we admit the affix "1" throughout and call the coordinate map $\xi = (T, Z)$ for $\xi \in \mathbf{R}_+ \times X$. The process sample path is then

$$x_t = \begin{cases} z_0 & t < T \\ Z & t \geq T. \end{cases}$$

As before $(\mathcal{F}_t)$ denotes the completed natural filtration of (x_t). It is not hard to see that $\mathcal{F}_t$ consists of all sets of the form $A \cap ([0,t] \times X)$ where $A \in \mathcal{F}$ together with $A_0(t) := (]t, \infty[\times X) \cup \{\Delta\}$ as an atom. (By an "atom" of the completed σ-field, we mean that if $A_0(t)$ is expressed as a disjoint union $A_0(t) = A_1 \cup A_2$ then $PA_1 = 0$ or $PA_2 = 0$.)

(1) Lemma Let τ be an $\mathcal{F}_t$-stopping time. Then there exists $t_0 \in [0, \infty]$ such that $\tau \wedge T = t_0 \wedge T$

Proof: This is a special case of Lemma (3.7).

□

For $A \in \mathcal{B}(X)$, define

$$F^A(t) := \mu(]t, \infty[\times A)$$

and let

$$F(t) := F^X(t) + \mu(\{\Delta\}) = P(T > t).$$

These are right-continous, decreasing functions. Now define

$$c := \inf\{t : F(t) = 0\},$$

so that $P(t \leq c) = 1$. We have to distinguish three cases

$$\begin{array}{llll} \text{case } 1: & c = \infty & & \\ \text{case } 2: & c < \infty & \text{and} & F(c-) = 0 \\ \text{case } 3: & c < \infty & \text{and} & F(c-) > 0 \end{array}$$

(Here and throughout $F(c-)$ denotes the left-hand limit:$F(c-) = \lim_{t \uparrow c} F(t)$.)

Any uniformly integrable (u.i.) martingale M_t of $\mathcal{F}_t$ takes the form $M_t = E[M_\infty | \mathcal{F}_t]$ for some integrable $\mathcal{F}_\infty$-measurable random variable M_∞. Here $\mathcal{F}_\infty = \mathcal{F}$ and hence all such random variables are of the form $M_\infty = h(T, Z)$ for some measurable function h satisfying

$$Eh(T,Z)| = \int_{]0,\infty[\times X} |h(t,z)| \mu(dt, dz) + |h(\Delta)| \mu(\{\Delta\}) < \infty$$

It will be notationally convernient to write the right-hand side of this expression as

$$\int\limits_{]0,\infty]\times X} |h| d\mu$$

and a similar convention will apply below to integrals over sets denoted $]t,c] \times X$ when $c = \infty$. One can then check from the definition of conditional expectation that the u.i. martingale M_t has the following expicit expression:

$$\begin{aligned} M_t &= E[h(T,Z)|\mathcal{F}_t] \\ &= I_{t\geq T} h(T,Z) + I_{t<T} \frac{1}{F(t)} \int\limits_{]t,c]\times X} h(t,z) h(dt,dz) \end{aligned} \tag{2}$$

A process (M_t) is a *local martingale* if there exists an increasing sequence τ_n of $\mathcal{F}_t$ stopping times such that $\tau_n \uparrow \infty$ a.s. and $M_t^n := M_{t\wedge\tau_n}$ is a u.i. martingale for each n.

(3) Theorem Let M_τ be a local martingale. Then

(a) M_τ is stopped at T, i.e. $M_\tau = M_{\tau\wedge T}$ a.s.

(b) In cases 1 and 2, M_t is a martingale on $[0,c[$

(c) In case 3, M_t is a u.i. martingale.

Proof: (a) Note from (2) above that any u.i. martingale is stopped at T. Hence if τ_n is a sequence of localizing times then

$$M_t = \lim_{n\to\infty} M_{t\wedge\tau_n} = \lim_{n\to\infty} M_{t\wedge\tau_n\wedge T} = M_{t\wedge T} \quad \text{a.s.}$$

(6) If $\tau_k \geq T$ as for some k then using (a) we have

$$M_{t\wedge\tau_k} = M_{t\wedge\tau_k\wedge T} = M_{t\wedge T} = M_\tau \quad \text{a.s.,}$$

so that M_t is a u.i. martingale. Thus suppose $P(\tau_k < T) > 0$ for all k. By Lemma (1) there is a sequence of real numbers t_k such that $\tau_k \wedge T = t_k \wedge T$ and we must have $t_k \uparrow c$ since $\tau_k \uparrow \infty$ a.s. Then

$$M_{t\wedge\tau_k} = M_{t\wedge\tau_k\wedge T} = M_{t\wedge t_k\wedge T} = M_{t\wedge t_k}$$

Thus M_τ is a u.i. martingale on $[0,t_k]$ and hence a martingale on $[0,c[$ since $t_k \uparrow c$.

(c) In case 3, $0 < F(c-) = P(T = c)$. Consider the sequence t_k as above. If $t_k < c$ for all k then on the set $(T = c)$ we have $\tau_k = t_k$ for all k, so that $\tau_k \not\to \infty$. Thus there must exist k' such that $t_{k'} = c$. But then $M_t = M_{t\wedge\tau_{k'}}$ so that M_t is a u.i. martingale. □

We now introduce the fundamental family of point processes associated with the jump process (x_t). For $A \in \mathcal{B}(X)$ and $t \geq 0$ define

$$\begin{aligned} p(t, A) &:= I_{t\geq T} I_{Z\in A} \\ \tilde{p}(t, A) &:= - \int\limits_{]0, T\wedge t]} \frac{1}{F(s-)} dF^A(s) \\ q(t, A) &:= p(t, A) - \tilde{p}(t, A). \end{aligned}$$

Note that the process $t \to p(t, A)$ has sample functions which are either identically zero, or have a unit jump at T if $T < t$ and $Z \in A$. We now show that $\tilde{p}$ is the "compensator"of p.

(4) Theorem For each $A \in \mathcal{B}(X)$, $\tilde{p}(t, A)$ is the unique predictable process such that the process $t \to p(t, A)$ is an $\mathcal{F}_t$-martingale.

Proof: $\tilde{p}(t, A)$ is clearly a predictable process in accordance with Definition (4.1). That it is a compensator follows by direct computation. Take $t > s$; then

$$E[p(t, A) - p(s, A)|\mathcal{F}_s] = I_{s\leq T} \frac{1}{F(s)} (F^A(s) - F^A(t)).$$

(Note that $p(t, A) - p(s, A) = 0$ if $s \geq T$.) On the other hand $\tilde{p}(t, A)$ is a function of T only, and $F(t)$ is the survivor function of T. Hence

$$\begin{aligned} E[\tilde{p}(t, A) - \tilde{p}(s, A)|\mathcal{F}_s] =& I_{s<T} \Big\{ - \frac{F(t)}{F(s)} \int\limits_{]s,t]} \frac{1}{F(u-)} dF^A(u) \\ &+ \frac{1}{F(s)} \int\limits_{]s,t]} \int\limits_{]s,r]} \frac{1}{F(u-)} dF^A(u) dF(r) \Big\}. \end{aligned}$$

Interchanging the order of integration, the second term on the right is

$$\begin{aligned} \frac{1}{F(s)} \int\limits_{]s,t]} \frac{1}{F(u-)} \int\limits_{[u,t]} dF(r) dF^A(u) &= \frac{1}{F(s)} \int\limits_{]s,t]} \frac{1}{F(u-)} (F(t) - F(u-)) dF^A(u) \\ &= \frac{F(t)}{F(s)} \int\limits_{]s,t]} \frac{1}{F(u-)} dF^A(u) + \frac{1}{F(s)} (F^A(t) - F^A(s)). \end{aligned}$$

Thus $E[\tilde{p}(t,A)-\tilde{p}(s,A)|\mathcal{F}_s] = E[p(t,A)-p(s,A)|\mathcal{F}_s]$ and hence $q(t,A) = p(t,A)-\tilde{p}(t,A)$ is a martingale, since both p and $\tilde{p}$ are $\mathcal{F}_t$-adapted processes. The predictable compensator is unique, by Theorem (4.5).

$\square$

We now want to consider "stochastic integrals" with respect to the family of martingales $q(t,A)$. These will simply be differences of ordinary (Stieltjes) integrals with respect to p and $\tilde{p}$, applied to suitable classes of integrands.

For p, the appropriate definition of the integral is clear. Identify $p(t,A)$ with a random set function ν on $\mathbf{R}_+ \times X$ such that

$$\nu([0,t] \times A) = p(t,A).$$

Then it is clear that ν is simply the Dirac measure $\delta_{(T,Z)}$ at (T,Z), since $\nu([0,t] \times A) = 1$ if $(T,Z) \in [0,t] \times A$ and $= 0$ otherwise. We therefore define, for any measurable function $g : Y \to R$

$$\int_Y g dp = \int_{]0,\infty] \times X} g(t,z) p(dx,dz) := g(T,Z).$$

Throughout, we will only consider functions g such that $g(\Delta) = 0$. Then we say that $g \in L_1(p)$ if

$$||g||_{L_1(p)} := E \int_Y |g| dp = E|g(T,Z)| < \infty.$$

Thus $L_1(p) = L_1(Y, \mathcal{Y}, \mu)$. We say that $g \in L_1^{loc}(p)$ if $g I_{(t<\tau_n)} \in L_1(p)$ for some sequence of stopping times $\tau_n \uparrow \infty$ a.s.

For $\tilde{p}$ we adopt a similar approach. We identify $\tilde{p}$ with the random set function $\tilde{\nu}$ defined by

$$\tilde{\nu}(]0,t] \times A) = \tilde{p}(t,A)$$

It is then easy to see that $\tilde{\nu}$ satisfies

$$\tilde{\nu}([0,t] \times A) = \int_{]0,t] \times A} I_{s \leq T} \frac{1}{F(s-)} \mu(ds,dz)$$

and therefore that $\tilde{\nu}$ coincides with the random measure

$$\tilde{\nu}(F) = \int_F I_{s \leq T} \frac{1}{F(s-)} \mu(ds,dz)$$

for $F \in \mathcal{Y}$. We thus define

$$\int_Y g d\tilde{p} := \int_Y g(s,z)\tilde{p}(ds,dz) : = \int_Y g(s,z)\tilde{\nu}(ds,dz)$$
$$= \int_{]0,\infty]\times X} g(s,z) I_{s\leq T} \frac{1}{F(s-)}\mu(ds,dz).$$

Again, we consider only functions g such that $g(\Delta) = 0$. The integral exists when $g \in L_1(\tilde{p})$ defined by

$$L_1(\tilde{p}) = \{g : Y \to R : ||g||_{L_1(p)} := E\int_Y |g| d\tilde{p} < \infty\}$$

and

$$L_1^{loc}(\tilde{p}) = \{g : gI_{s<\tau_n} \in L_1(\tilde{p}) \quad \text{for stopping times} \quad \tau_n \uparrow \infty \text{ a.s.}\}$$

(5) Proposition $L_1(p) = L_1(\tilde{p})$ and $L_1^{loc}(p) = L_1^{loc}(\tilde{p})$. Also

(6) $||g||_{L_1(\tilde{p})} = ||g||_{L_1(\tilde{p})}$

Proof: We need only show (6). Note again that $\int g d\tilde{p}$ is a function only of T, whose survivor function is F. Hence

$$||g||_{L_1(\tilde{p})} = -\int_{]0,\infty]} \int_Y I_{s\leq t} |g(s,x)| \frac{1}{F(s-)}\mu(ds,dx) dF(t)$$
$$= -\int_Y |g(s,x)| \frac{1}{F(s-)} \Big(\int_{[s,\infty]} dF(t)\Big)\mu(ds,dz)$$
$$= \int_Y |g(s,x)|\mu(ds,dz) = ||g||_{L_1(p)}$$

□

(7) Proposition $L_1^{loc}(p) = L_1^{loc}(d\mu)$ where

$$L_1^{loc}(d\mu) = \{g : Y \to \mathbf{R} : \ gI_{s\leq t} \in L_1(Y, \mathcal{Y}, \mu) \quad \text{for all} \quad t < c\}$$

Proof: Suppose $g \in L_1^{loc}(p)$, let τ_n be a sequence of localizing times and let t_n be the associated sequence of constants such that $\tau_n \wedge T = t_n \wedge T$. Then

$$\int_Y gI_{t<\tau_k} dp = g(T,Z) I_{T<\tau_k}$$

and $(T < \tau_k) =]0, t_k[\times X$. Hence $g \in L_1^{loc}(d\mu)$, since $t_k \uparrow c$. Conversely, if $g \in L_1^{loc}(d\mu)$, take any sequence $t_k \uparrow c$ and introduce the following stopping times:

$$\begin{aligned} c = \infty: &\qquad \tau_k = k \\ c < \infty, F(c-) = 0: &\quad \tau_k = kI_{T \le t_k} + t_k I_{T > t_k} \\ c < \infty, F(c-) > 0: &\quad \tau_k = \infty \end{aligned}$$

Then $\tau_k \uparrow \infty$ a.s. and it is easily shown that $gI_{t \le \tau_k} \in L_1(p)$.

□

For $g \in L_1^{loc}(p)$ we can now define a process (M_t^g) by

$$\begin{aligned} M_t^g := &\int_Y I_{s \le t} g(s,z) q(ds,dz) \\ = &\int_Y I_{s \le t} g(s,z) p(ds,dz) - \int_Y I_{s \le t} g(s,z) \tilde{p}(ds,dz). \end{aligned}$$

M_t^g is given more explicitly, from the definition, as follows

$$M_t^g = g(T,Z) I_{t \ge T} + \int_{]0,T \wedge t] \times X} g(s,z) \frac{1}{F(s-)} \mu(ds,dz). \tag{8}$$

The following result is then proved by direct calculations similar to those in the proof of Theorem (4) above.

(9) Theorem (M_t^g) is a martingale for $g \in L_1(p)$, and a local martingale for $g \in L_1^{loc}(p)$

Suppose M_t is a uniformly integrable $\mathcal{F}_t$-martingale; then M_t takes the form $M_t = E[M_\infty | \mathcal{F}_t]$ for some $\mathcal{F}$-measurable random variable M_∞ such that $E|M_\infty| < \infty$. However, all such random variables can be written as $M_\infty = h(T,Z)$ for some measurable function h on Y such that $E|h(T,Z)| < \infty$. Then

$$\begin{aligned} M_t &= E[h(T,Z) | \mathcal{F}_t] \\ &= I_{t \ge T} h(T,Z) + I_{t < T} \frac{1}{F(t)} \int_{]t,\infty] \times X} h \, d\mu \end{aligned}$$

If $M_0 = 0$ a.s. then $Eh(T,Z) = 0$, i.e.

$$0 = \int_{]0,t] \times X} h \, d\mu + \int_{]t,\infty] \times X} h \, d\mu,$$

so every u.i. martingale such that $M_0 = 0$ takes the form

$$M_t = I_{t \ge T} h(T,Z) - I_{t < T} \frac{1}{F(t)} \int_{]0,t] \times X} h \, d\mu. \tag{10}$$

We want to show that $M_t = M^g_\tau$ for some integrand g. To get an idea what g must be, consider the following example

(11) Example Take $X = \mathbf{R}$ and suppose $\mu(ds, dx) = \Psi(s,x)dsdx$ for some density function Ψ. Then from (8), for $g \in L_1(p)$

$$
\begin{aligned}
M^g_t =& I_{t\geq T}\big(g(T,Z) - \int_0^T \int_R \frac{1}{F(s)} g(s,x)\Psi(s,x)dxds\big) \\
&+ I_{t<T} \int_0^t \int_R \frac{1}{F(s)} g(s,x)\Psi(s,x)dxds.
\end{aligned} \tag{12}
$$

If M_t is a martingale with associated function h, then $M_t = M^g_t$ only if the coefficients of $I_{t\geq T}$ in (10),(12) agree, i.e.

$$
h(t,z) = g(t,z) - \int_0^t \int_{\mathbf{R}} \frac{1}{F(s)} g(s,x)\Psi(s,x)dxds. \tag{13}
$$

Define $\eta(t) = g(t,z) - h(t,z)$ (noting that it does not depend on z) and

$$
f(s) = \int_{\mathbf{R}} \Psi(s,x)dx, \quad \gamma(s) = \int_{\mathbf{R}} h(s,x)\Psi(s,x)dx.
$$

Then from (13) we have

$$
\eta(t) = \int_0^t \int_{\mathbf{R}} \frac{1}{F(s)}(\eta(s) + h(s,x))\Psi(s,x)dxds = \int_0^t \frac{f(s)}{F(s)}\eta(s)ds + \int_0^t \frac{1}{F(s)}\gamma(s)ds.
$$

Thus $\eta(t)$ satisfies the linear ordinary differential equation

$$
\frac{d}{dt}\eta(t) = \frac{f(t)}{F(t)}\eta(t) + \frac{1}{F(t)}\gamma(t), \quad \eta(0) = 0
$$

whose unique solution is

$$
\eta(t) = \int_0^t \exp\left(\int_s^t \frac{f(u)}{F(u)}du\right)\frac{1}{F(s)}\gamma(s)ds = \frac{1}{F(t)}\int_0^t \gamma(s)ds,
$$

where the last equality follows from the fact that $f(s) = -dF(s)/ds$. This shows that the coefficients of $I_{t\geq T}$ in (10),(12) agree if

$$
g(t,z) = h(t,z) + \frac{1}{F(t)}\int_0^t \int_{\mathbf{R}} h(s,x)\Psi(s,x)dxds
$$

It is easily shoved that with this choice of g the coefficients of $I_{t\leq T}$ in (10),(12) agree as well, so $M_t = M_t^g$.

The general result is as follows

(14) Theorem (M_t) is a local martingale of $\mathcal{F}_t$ with $M_0 = 0$ if and only if $M_t = M_t^g$ for some $g \in L_1^{loc}(p)$.

Proof: We have already shown that M_t^g is a local martingale for $g \in L_1^{loc}(p)$. Thus suppose that M_t is a local martingale with $M_0 = 0$. We have two cases:

Case 1: $c < \infty$, $F(c-) > 0$: It was shown above that M_t is then a u.i. martingale, and hence of the form (10) for some h with $\int |h| d\mu < \infty$. Consider the function g given by

$$g(t,z) = h(t,z) + I_{t<c}\frac{1}{F(t)}\int_{]0,t]\times X} h(s,x)\mu(ds,dx). \tag{15}$$

We can verify by direct calculation that (a) $M_t = M_t^g$ for $t < c$, and (b) M_t and M_t^g are stopped at c and $M_c = M_c^g$ when $T(\omega) = c$. Now

$$\begin{aligned}
||g||_{L_1(p)} = \int_Y |g| d\mu &\leq \int_Y |h| d\mu - \int_{]0,c[} \frac{1}{F(t)} \int_{]0,t]\times X} |h(s,x)|\mu(ds,dx)dF(t) \\
&\leq \int_Y (h) d\mu - \frac{1}{F(c-)} \int_{]0,c[} \int_{]0,t]\times X} |h| d\mu dF(t) \\
&= \int_Y |h| d\mu + \frac{1}{F(c-)} \int_Y (F(s) - F(c-))|h| d\mu \\
&\leq (1 + \frac{1}{F(c-)})||h||_{L_1(d\mu)}.
\end{aligned}$$

Thus $g \in L_1(p)$.

Case 2: $c = \infty$ or $c < \infty, F(c-) = 0$. Here M_t is a u.i. martingale on $[0,r]$ for any $r < c$, and hence of the form (10) for some h satisfying

$$\int_{]0,r]\times X} |h| d\mu < \infty \qquad \text{for all} \quad r < c$$

Calculations as before show that $M_t = M_1^g$ with g given by (15), and for $r < c$

$$\begin{aligned}
\int_{]0,r]\times X} |g| d\mu &\leq \int_{]0,r]\times X} |h| d\mu - \int_{]0,r]} \frac{1}{F(t)} \int_{0,t]\times X} |h| d\mu dF(t) \\
&\leq (1 + \frac{1}{F(r)}) \int_{]0,r]\times X} |h| d\mu
\end{aligned}$$

This shows that $g \in L_1^{loc}(p)$ in view of Proposition (7).

□

(16) Remark The following extension of the preceding results will be needed in the next section. Suppose that $(\Xi, \mathcal{S}, m)$ is a probability space and define $\Omega = \Xi \times Y$. Let $\mathcal{F}_0 = \mathcal{S} \times Y$ and $\mathcal{F}_t^0 = \mathcal{F}_0 \vee \sigma\{x_s, s \leq t\}$ where the path x_t is defined as before except that z_0 is now an $\mathcal{F}_0$-measurable random variable let $\mu(\xi; dt, dz)$ be the conditional measure of (T, Z) given $\mathcal{F}_0$, so that a probability measure P on Ω is defined by

$$P(A_1 \times A_2) = \int_{A_1} \mu(\xi; A_2)\mu(d\xi) \qquad A_1 \in \mathcal{S},\ A_2 \in \mathcal{Y}.$$

Let $\mathcal{F}_t$ be the P-completion of $\mathcal{F}_t^0$. Then the characterization of $\mathcal{F}_t$-local martingales given in Theorem (14) remains unchanged, except for the obvious modifications to the class of integrands g to allow for ξ-dependence.

A.6 Local Martingale representation for the general jump process

We now revert to consideration of the general multi-jump process as described in §A.2. We define the family of elementary point processes $p(t, A)$ for $t \geq 0, A \in \mathcal{B}(X)$ as

$$p(t, A) = \sum_k I_{t \geq T_k} I_{Z_k \in A},$$

and define

$$F^{A,1}(u) = \mu^1(]u, \infty[\times A) \qquad F^1(u) = f^{X,1}(u) + \mu^1(\{\Delta\})$$

and for $k = 2, 3, ..$

$$f^{A,k}(\omega_{k-1}, h) = \mu^k(\omega^{k-1};]u, \infty[\times A) \qquad F^k(\omega_{k-1}, u) = F^{X,k}(\omega_{k-1}, u) + \mu^k(\omega_{k-1}; \{\Delta\})$$

$$\Phi_k^A(\omega_{k-1}, s) = -\int_{]0,s]} \frac{1}{F^k(\omega_{k-1}, u-)} dF^{A,k}(\omega_{k-1}, u).$$

Let

$$\tilde{p}(t, A) = \Phi_1^A(S_1) + .. + \Phi_{k-1}^A(\omega_{k-2}, S_{k-1}) + \Phi_k^A(\omega_{k-1}, t - T_{k-1}) \qquad \text{for} \quad t \in]T_{k-1}, T_k]$$

This is a predictable process.

(1) Proposition For $A \in \mathcal{B}(X)$, let

$$g(t, A) = p(t, A) - \tilde{p}(t, A)$$

then for each fixed k, A, the process $h(t \wedge T_k, A)$ is an $\mathcal{F}_t$-martingale, i.e. $\tilde{p}(t, A)$ is the predictable compensator of $p(t, A)$.

Proof: This is proved by direct calculation.

$\square$

An integrand g for stochastic integration is made up from functions g^k in the following way, where for each $a = 1, 2, .. \; g^k : \Omega_k \to R$ is a measurable function such that $g(\omega_{k-1}; \Delta) = 0$ for all ω_{k-1}.

$$(2) \qquad g(\omega, t, z) = \begin{cases} g^1(t, z) & t \leq T_1(\omega) \\ g^k(\omega_{k-1}; t, z) & T_{k-1}(\omega) < t \leq T_k(\omega) \\ 0 & t \geq T_\infty(\omega) \end{cases}$$

Equivalently, g is a measurable function such that for each $z \in X$ the map $(t, \omega) \to g(\omega, t, z)$ is a predictable process. The integrals of g with respect to p and $\tilde{p}$ are defined in a way which directly generalizes the Definitions in §A.4, namely

$$\int_\Omega g dp = \sum_{k=1}^{\infty} g^k(S_1, Z_1, .., S_k, Z_k)$$

$$\int_\Omega g d\tilde{p} = \int_{Y_1} I_{s \leq S_1} g^1(s, z) \frac{1}{F^1(s-)} \mu^1(ds, dz)$$

$$+ \sum_{k=2}^{\infty} \int_{Y_k} I_{s \leq S^k} g^k(\omega_{k-1}, s, z) \frac{1}{F^k(\omega_{k-1}, s)} \mu^k(\omega_{k-1}; ds, dz).$$

Note that these are finite sums if $S_k = \infty$ for some k. The definitions of $L_1(p), L_1^{loc}(p)$ etc. read exactly as before, except that the localizing times τ_n are assumed to converge to T_∞, not ∞.

(3) Proposition Suppose $g \in L_1^{loc}(p)$ and define

$$M_t^g = \int_{]o,t] \times X} q(\omega, s, z) g(ds, dz)$$

where $q = p - \tilde{p}$. Then there exists a sequence of stopping times τ_n such that $\tau_n < T_\infty, \tau_n \uparrow T_\infty$ and $M_{t \wedge \tau_n}^g$ is a u.i. martingale for each n.

Proof: Take $\tau_n = \sigma_n \wedge T_n$ where σ_n are localizing times for g, i.e. $g I_{t < \sigma_n} \in L_1(p)$ for each n. A direct calculation shows that $M_{t \wedge \tau_n}^g$ is a martingale, and $M_{t \wedge \tau_n}^g = E[M_{\tau_n}^g | \mathcal{F}_{t \wedge \tau_n}]$, showing that $M_{t \wedge \tau_n}^g$ is u.i. $\square$

Let M_t be a u.i. $\mathcal{F}_t$-martingale. Then $M_t = E[M_\infty|\mathcal{F}_t]$ for some $M_\infty \in \mathcal{F}_\infty = \mathcal{F}$ (Lemma (2.2)). From theorem (3.1) we know that $\mathcal{F} = \bigvee_n \mathcal{F}_{T_n}$; hence

$$(4) \qquad M_{T_\infty -} = lim_{n\to\infty} M_{T_n} = lim_{n\to\infty} E[M_\infty|\mathcal{F}_{T_n}] = E[M_\infty|\mathcal{F}] = M_\infty.$$

Thus any u.i. martingale is stopped at T_∞ and is left-continous there. The same therefore applies to local martingales.

We now come to the main result.

(5) Theorem Let M_t be a local $\mathcal{F}_t$-martingale. Then $M_t = M_t^g$ for some $g \in L_1^{loc}(p)$

Proof: First, suppose that M_t is u.i. We can then express M_t in the form

$$(6) \qquad M_t = M_{t\wedge T_1} + \sum_{k=2}^{\infty}(M_{t\wedge T_k} - M_{T_{k-1}})I_{t\geq T_{k-1}}.$$

Indeed, this is an identify if $t < T_\infty$ and the right-hand side is equal to $lim_k M_{T_k} = M_{T_\infty -}$ if $t \geq T_\infty$; from (3), $M_t = M_{T_\infty -}$ in this case. Define

$$(7) \qquad X_t^k := M_{(t+T_{k-1})\wedge T_k} - M_{T_{k-1}}, \quad t \geq 0.$$

Then

$$(8) \qquad M_{t\wedge T_k} - M_{T_{k-1}} = X_{(t-T_{k-1})\vee 0}$$

and X_t^k is a u.i. martingale with respect to the filtration $\mathcal{H}_t = \mathcal{F}_{(t+T_{k-1})\wedge T_k}$. From Theorem (3.1) we know that $\mathcal{H}_t = \mathcal{F}_{T_{k-1}} \vee \sigma\{x_{(s+T_{k-1})\wedge T_k}, s \in [0,t]\}$, and thus X_t^k takes the form $X_t^k = E[h^k(\omega_{k-1}; S_k, Z_k)|\mathcal{H}_t]$.

Since $E|X_t^k| < \infty$ we have

$$\int_{\Omega_{k-1}} \int_{Y} |h^k(\eta; s, z)|\mu^k(\eta; ds, dz)\nu^{k-1}(d\eta) < \infty$$

where ν^k is the martingale distribution of ω_{k-1}. Thus from the 1-jump result, Theorem (4.14), and Remark (4.16), we can represent X_t^k as

$$X_t^k = \int_{]0,t]\times X} g^k(\omega_{k-1}; s, z) q^k(ds, dz),$$

where $q^k(t,A) := q((t+T_{k-1})\wedge T_k, A)$, for some integrand g^k satisfying

$$(9) \qquad \int_{]0,r]\times X} |g^k| d\mu^k \leq \left(1+\frac{1}{F^k(\omega_{k-1},r)}\right) \int_{]0,r]\times X} |h^k| d\mu^k$$

for all $r < c^k(\omega_{k-1}) := inf\{t : F^k(\omega_{k-1},t) = 0\}$. The collection g^k defines an integrand g such that $M_t = M_t^k$ a.s. for each t; it remains to show that $g \in L_1^{loc}$.

For $n = 1, 2..$, define $S_n^k(\omega_{k-1})$ as follows (omitting ω_{k-1}-dependence for convenience):

if $c^k = \infty$ or $c^k < \infty$ and $F^k(c^k-) \leq \frac{1}{n^3}$: $S_n^k := inf\{t : F^k(t) \leq \frac{1}{n^3}\}$

if $c^k < \infty$ and $F^k(c^k-) \geq \frac{1}{n^3}$: $S_n^k := c^k$

Then, from (9)

$$(10) \qquad \int_{\Omega_{k-1}} \int_Y I_{S<S_n^k} |g^k| d\mu^k d\nu^{k-1} \leq (1+n^3) \int_{\Omega_{k-1}} \int_Y |h^k| d\mu^k d\nu^{k-1} < \infty$$

Now define $\tau_n := T_j + S_n^j$ where $j := min\{k : T_k + S_n^k \leq T_{k+1}\}$. Then τ_n is an $\mathcal{F}_t$-stopping time, and

$$P[\tau_n < T_n] \leq P \bigcup_{k=1}^{n} (S_n^k < S_k) \leq n\frac{1}{n^3} = \frac{1}{n^2}$$

Thus $\sum P[\tau_n < T_n] < \infty$, so that by the Borel-Cantelli Lemma, $P[lim\ inf(\tau_n \geq T_n)] = 1$. It follows that $\tau_n \uparrow T_\infty$ a.s. Now

$$\int_Y |g| I_{t\leq T_n} dp = |g(T_1, Z_1)| + .. + |g^n(T_1, Z_1, ..T_{n-1}, Z_{n-1}; S_n Z_n)|,$$

so, using (10)

$$E \int_Y I_{t\wedge\tau_n\wedge T_n} |g| dp \leq \sum_{k-1}^{n} \int_{\Omega_{k-1}} \int_Y I_{S<S_n^k} |g^k| d\mu^k d\nu^{k-1} < \infty.$$

Since $\tau_n \wedge T_n \uparrow T_\infty$, this shows that $g \in L^1_{loc}(\rho)$, as claimed.

□

(11) Remarks (i) When $T_\infty = \infty$ a.s., Proposition (3) and Theorem (5) assert that M_t is a local $\mathcal{F}_t$-martingale if and only if $M_t = M_t^g$ for some $g \in L_1^{loc}(\rho)$.
(ii) The situation is slightly unsatisfying in that we have shown $g \in L_1(p)$ is a sufficient condition for M_t^g to be a martingale, but we have not shown that this condition is necessary.

The same point arises in connection with Ito stochastic integrals: if B_t is a Brownian motion and Ψ_t a nonanticipative integrand then the Ito integral $\int_0^t \Psi_s dB_s$ is defined and is a local martingale when $\int_0^t \Psi_s^2 ds < \infty$ a.s. for all t. The integral is a martingale when $E\int_0^t \Psi_s^2 ds < \infty$ for all t, but again this is only a *sufficient* condition.

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